Growth Rate of the Binomial Coefficient

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1 Computing $C(n,k)$

The number of additions made by the recursive algorithm for computing $C(n,k)$ based on the formula $C(n,k) = C(n-1,k-1) + C(n-1,k)$ is equal to $C(n,k) - 1$, which can be said to be $\Theta(C(n,k))$. But, what is the growth rate of $C(n,k)$? Specifically, how is $C(n,k)$ expressed in terms of the basic asymptotic efficiency classes?

2 Growth Rate of $C(n,k)$

To express the growth rate of $C(n,k)$ in terms of n, we consider the function $C(n,k)$ for different values of k.

For $k = 1$,

$$C(n,1) = n \in \Theta(n).$$

For $n = 2$,

$$C(n,2) = \frac{n(n-1)}{2} \in \Theta(n^2).$$

For $n = 3$,

$$C(n,3) = \frac{n(n-1)(n-2)}{6} \in \Theta(n^3).$$

For $n = \frac{n}{2}$,

$$C(n,n/2) = \frac{n!}{\Big((n/2)!\Big)^2}.$$

Using Stirling formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, as $n \to \infty$, we get,

$$C(n,n/2) = \frac{\sqrt{2\pi n}(n/e)^n}{[\sqrt{2\pi n/2}(n/2e)^{n/2}]^2} = \frac{2}{\sqrt{2\pi n}}2^n \in \Theta(2^n n^{-0.5}) \tag{1}$$

Equation 1 implies that $C(n,k)$ grows very fast for k close to $n/2$